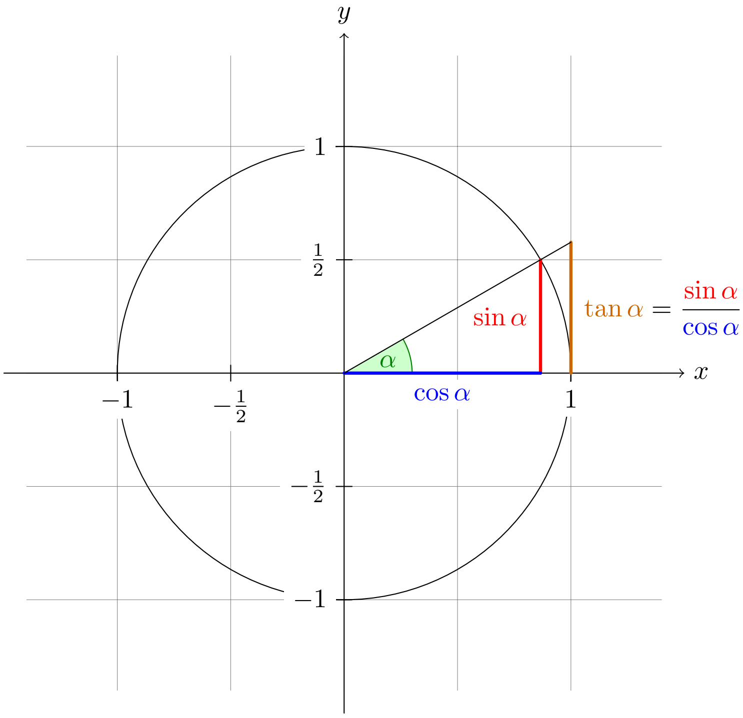
0.1 Trigonometry



The angle α is 30° in the example ($\pi/6$ in radians). The sine of α, which is the height of the red line, is

$$\sin\alpha = 1/2.$$

By the Theorem of Pythagoras we have $\cos^2\alpha + \sin^2\alpha = 1$. Thus the length of the blue line, which is the cosine of α, must be

$$\cos\alpha = \sqrt{1 - 1/4} = \tfrac{1}{2}\sqrt{3}.$$

This shows that $\tan\alpha$, which is the height of the orange line, is

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = 1/\sqrt{3}.$$

Table 1: Special angles and their trigonometric value

°	0	30	45	60	90
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

The inverse of trigonometric functions are called cyclometric functions (Atan,Asin,Acos,Acotg). Here the inverse question is asked meaning, you are given a real number and you have to calculate the angle that corresponds with that number. Also in this case special values are to be known. You need to be carefull with calculating the angle because you have always an angle modulo. (cf $sin(x) = sin(180° - x)$)

In the case of Asin and Acos you can find them in table 1. In case of Atan you need to know

Table 2: Special cyclometric values for Atan

Atan	°
0	0 **or** 180
1	45
$+\infty$	90

For $Atan\alpha = 0$ you need to be very carefull.The value of your angle depends on the way you found this result during your calculations.

0.2 Complex numbers

If you want to calculate the square root of a negative real number you can't find a real number.It simply doesn't exist. Basically a real number, negative or positive, to the second power is always positive.

For this reason we need to define a new kind of number to calculate the square root of a negative real number and these are called complex numbers.

By definition we say

$$\sqrt{-1} = j$$

or

$$j^2 = -1$$

In general a complex number is written as

$$a + bj$$

in which a is called the real part and b is called the complex part. We notice immediately that a complex number has two dimensions, so if we want to draw this number on somekind of a scheme we need to do this in two dimensions, in other words a plane. We cannot draw this kind of number on a straight line as we did with the real numbers. We draw the real part a (Re) on the X-axis and the complex part b (Im) on the Y-axis.

If we look at the drawing of an arbitrary complex number $a + bj$ in the complex plane we see that we can use the basic theory of trigonometry. In general if we calculate the argument ϕ and module M we use

$$M = \sqrt{(Re)^2 + (Im)^2}$$

and

$$\phi = Atan\frac{Im}{Re}$$

It is of paramount importance that if you calculate the argument you check the correct value of your angle. One has for the same complex number two possible solutions for the argument and just one is correct. To arrive at the correct value of the argument you need to look at the signs of your different parts (Re and Im)in your complex number. For example

Table 3: Values for the argument of a+bj

a+bj	°
1+j	$Atan\frac{1}{1} = Atan(1) = 45$
1-j	$Atan\frac{-1}{1} = Atan(-1) = -45$
-1+j	$Atan\frac{1}{-1} = Atan(-1) = -45$
-1-j	$Atan\frac{-1}{-1} = Atan(1) = 45$

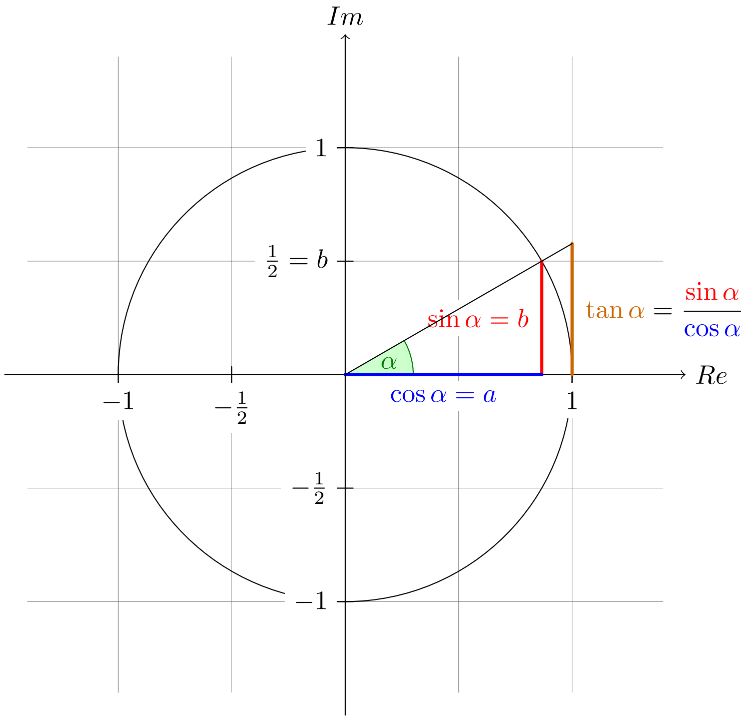
This is completely wrong!!

You have to look at the signs of respectively complex and real part and look at the quadrant they are located at in the complex plane!!

The correct solution becomes

Table 4: Values for the argument of a+bj

a+bj	°
1+j	$Atan\frac{1}{1} = Atan(1) = 45$
1-j	$Atan\frac{-1}{1} = Atan(-1) = -45$
-1+j	$Atan\frac{1}{-1} = Atan(-1) = 45 + 90 = 135$
-1-j	$Atan\frac{-1}{-1} = Atan(1) = -45 - 90 = -135$



We take the same example as in the explanation of the trigonometric functions. The sine of α, which is the lenght of the red line, is also the module of the complex part b of our complex number.The cos of α, which is the lenght of the blue line, is the module of the real part a of the complex number.
Very usefull is calculating module and argument of this complex number.

$$M = \sqrt{a^2 + b^2}$$

$$\phi = Atan\frac{b}{a}$$

Very important is noticing that a number with a complex part in the nominator is very difficult to use, so we need to avoid having a complex number in the nominator. We manage this by multiplying the nominator with its conjugate(see 0.4). The conjugate of a complex number is

$$a + bj \leftrightarrow a - bj$$

0.3 Basics on polynomes

0.3.1 Bezout's theorem

A polynome of the nth order has n roots in the complex plane.

0.3.2 special products of polynomes

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

so

$$(a + bj)(a - bj) = a^2 + b^2$$

Remember this last formula because you will use it very often when doing calculations during the lectures of automatisation.

0.4 Calculation with fractions

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} * \frac{d}{c}$$

$$\frac{1}{a} + \frac{1}{b} \neq \frac{1}{a+b}$$

but

$$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{a*b}$$

You make sure you put each fraction in the sum on the same nominator.

The following technique is used frequently in automatisation and is used to avoid having a complex number in the nominator.

$$\frac{1}{a+bj} = \frac{1}{a+bj} * \frac{a-bj}{a-bj} = \frac{a-bj}{a^2+b^2}$$

0.5 exercise

Calculate module and argument of $\frac{1}{3+5j}$

$$\frac{1}{3+5j}\frac{3-5j}{3-5j} = \frac{3-5j}{3^2+5^2} = \frac{3-5j}{34}$$

$$M = \sqrt{(\frac{3}{34})^2 + (\frac{5}{34})^2} = \sqrt{\frac{34}{34^2}} = \sqrt{\frac{1}{34}}$$

$$\phi = Atan\frac{\frac{-5}{34}}{\frac{3}{34}} = Atan\frac{-5}{3} = -59^\circ$$

Our complex number lies in the 4th quadrant so the choice of angle was correct. If the complex number would have been $\frac{1}{-3-5j}$ it would have been $(-59-90)^\circ$.(Check this out for yourself)

0.6 Special functions and some of their special values

$$log(a^b) = b * log(a)$$

$$log(a*b) = log(a) + log(b)$$

$$log(a+b) \neq log(a) + log(b)$$

$$a^0 = 1$$

Table 5: Special values for log

a	log(a)
0	-∞
+∞	+∞
1	0

0.7 Laplace transforms and differential equations

exercise: serial RLC circuit

An inductor of 2H, an resistance of 16 ohm and a capacitance of 0.02F are connected in series with a source of 300V DC. Calculate the current at any time $t \geq 0$ if at t=0 the charge on the capacitor and the current in the circuit are zero.

solution:

a)inductor: using Lenze's law: $L\frac{di}{dt} = U_L$
b)resistance:using Ohm's law:$Ri = U_R$
c)capacitance:using Farad's law:$\int \frac{i}{C}\,dt = U_C$
Our model becomes $U = U_L + U_R + U_C$

$$L\frac{di}{dt} + Ri + \int \frac{i}{C}\,dt = U$$

$$2\frac{di}{dt} + 16i + \int \frac{i}{0.02}\,dt = 300$$

Because we have information about the charge on the capacitor we rewrite the differential equation in function of charge and not current. We know $i = \frac{dq}{dt}$ so

$$2\frac{d^2q}{dt^2} + 16\frac{dq}{dt} + \frac{q}{0.02} = 300$$

Taking Laplace transform

$$(2s^2q - sQ(0) - PQ'(0)) + 8(sq - Q(0)) + 50q = \frac{300}{s}$$

We know $Q(0) = 0$ and $I(0) = Q'(0) = 0$ thus

$$(2s^2q) + 8(sq) + 50q = \frac{300}{s}$$

$$q = \frac{300}{s(2s^2 + 16s + 50)}$$

After working out in partial fractions we get

$$q = \frac{6}{s} + \frac{6(s+4)}{(s+4)^2 + 9} + \frac{24}{(s+4)^2 + 9}$$

Taking the inverse Laplace transform

$$q = 6 - 6e^{-4t}\cos 3t - 8e^{-4t}\sin 3t$$

$$I = \frac{dq}{dt} = 50e^{-4t}\sin 3t$$

0.7.1 Properties of Laplace transforms

1. transform of a differential:
$$\mathcal{L}(\frac{df(t)}{dt}) = sF(s) - f(0)^1$$
2. transform of an nth order differential:
$$\mathcal{L}(F^{(n)}(t)) = s^n f(s) - s^{(n-1)}F(0) - s^{n-2}F'(0) - ... - F^{(n-1)}(0)$$
3. transform of an integral:
$$\mathcal{L}(\int f(t)\,dt) = \frac{F(s)}{s}$$

[1] We suppose that f(0)=0

4. final value theorem:

$$\lim_{t \to +\infty} f(t) = \lim_{s \to 0} sF(s).$$

5. initial value theorem:

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} F(s)$$

6. time shift:

$$f(t-a) = e^{-as} F(s)$$